ON DIRECTED GRAPHS AND RELATED TOPOLOGICAL SPACES

by

Thomas J. Ahlborn

UNPUBLISHED PRELIMINARY DATA

~~TECHNICAL REPORT NO. 2~~

PREPARED UNDER RESEARCH GRANT NO. NsG-568
(PRINCIPAL INVESTIGATOR: T. N. BHARGAVA)
FOR

NATIONAL AERONAUTICS and SPACE ADMINISTRATION

DEPARTMENT OF MATHEMATICS

KENT STATE UNIVERSITY

KENT, OHIO

September 30, 1964

ABSTRACT

ON DIRECTED GRAPHS AND RELATED TOPOLOGICAL SPACES.

33402

In this report, we investigate certain tie-ups between the theory of directed graphs and point-set topology. This work extends certain aspects of the work done by Bhargava in "A Stochastic Model for Time Changes in a Binary Dyadic Relation".

With each directed graph $\Gamma(A,E)$ on an arbitrary set A, we associate a unique topological space $(A,\mathfrak{T})$ by defining a set $\underline{a} \subseteq A$ to be open if there does not exist an edge in $\Gamma(A,E)$ from set $(A \sim \underline{a})$ to set $\underline{a}$; each such topology is shown to have the property of completely additive closure. We obtain several theorems relating connectedness and accessibility properties of a directed graph to properties of the topology determined by that directed graph. It is found that the connectedness of a directed graph is, in a certain sense, consistent with the "topological connectedness" of the topological space determined by that directed graph. We further investigate these topologies in terms of the closure, kernel, and core operators.

We show that the definition of an open set, as given here, establishes a single valued mapping of the family of all directed graphs on set A onto the family of all topologies with completely additive closure on set A. This mapping also maps one-to-one the family of all transitive directed graphs with loops on set A onto the family of all

topologies with completely additive closure on set A. Furthermore, the transitive directed graph with loops mapped to a particular topology is, in each case, the directed graph with the maximum edge set determining that topology. On the other hand, we find that there does not necessarily exist a directed graph with a minimal edge set determining a particular topology.

Finally we make a brief study of the properties of topologies obtained from a directed graph with respect to two other definitions for an open set.

Acknowledgments

With deep appreciation, I sincerely thank Dr. T. N. Bhargava for his inspiration and patient guidance during the development and preparation of this report.

Also, I gratefully acknowledge the interest and helpful suggestions of Drs. C. E. Aull (Kent State University), S. D. Chatterji (Mathematics Research Center, U. S. Army, University of Wisconsin), and P. H. Doyle (Virginia Polytechnic Institute).

I express thanks to the National Aeronautics and Space Administration for financial support during the development of this report through their grant number Ns G-568.

TABLE OF CONTENTS

CHAPTER O.

INTRODUCTION

The object of this report is to investigate certain tie-ups between the theory of directed graphs and point-set topology. In a certain sense this report carries further some of the work done by Bhargava in [3], where he develops a probabilistic model for the study of binary dyadic relations. An aggregate of binary dyadic relations on a set has two isomorphic representations: (i) a directed graph, and (ii) an incidence matrix. Applications of mathematical models of these types have been treated by Bhargava[1] and are being further investigated[2] by him. In this report, we do not consider applications, rather, we study certain theoretical aspects of directed graphs in terms of point-set topology.

The notions of accessibility of points and of connectedness of a directed graph appear similar respectively to the concepts of a closed set and of connected sets as used in point-set topology. Therefore, it seems natural, as we have done in this report, to investigate possible tie-ups between directed graphs and point-set topology. It is hoped that the results of this work will be helpful in the study and analysis of mathematical models similar to those developed in [3].

---

[1] With L. Katz, "A Stochastic Model for a Binary Dyadic Relation with Applications to Social and Biological Sciences", Bull. Inst. Internat. Statist. 40 (1964), 1055-1057.

[2] Under National Aeronautics and Space Administration, research grant number Ns G-568.

A directed graph (or simply a digraph) consists of a set of points and a set of edges (possibly void) joining particular ordered pairs of points, whereas a topology is a family of sets in which the intersection of any two sets is a member of the family and the union of the members of each subfamily is a member of the family.

Our investigation consists of establishing a topology on an arbitrary digraph and then expressing the notions of accessibility of points and connectedness of a digraph in terms of that topology. We relate certain topological concepts, for example the separation axioms and the closure operator, to some of the connectedness properties of digraphs. We also make a comparative study of the family of digraphs with respect to the family of topologies determined by these digraphs and briefly consider other topologies which may be established on a digraph.

CHAPTER I.

PRELIMINARIES

SECTION 1.1. DEFINITIONS AND NOTATIONS

The following definitions and notations serve as a basis for our discussion of directed graphs throughout this report. Most of these are standard and have appeared elsewhere (e.g., see Bhargava [3]).

Let A denote a set of points, $A = \{ A_i : i \in \mathcal{I}$, an index set$\}$. Set A may be a finite, a countable, or an uncountable set. We denote a subset of A by $\underline{a}$.

Let ϵ denote a subset of the cartesian product A X A, $\epsilon \subseteq A \times A$.

DEFINITION 1.1.1. A digraph (directed graph) is a set A of points and a set E of ordered pairs of points such that $\phi \subseteq E \subseteq A \times A$. A digraph is denoted by $\Gamma(A,E)$, or simply by $\Gamma(A)$ if the set E is fixed.

DEFINITION 1.1.2. For $\underline{a} \subseteq A$, the digraph $\Gamma(\underline{a}, E \cap \underline{a} \times \underline{a})$, denoted simply by $\Gamma(\underline{a})$, is a subdigraph of the digraph $\Gamma(A,E)$.

DEFINITION 1.1.3. An element of E is called an <u>edge</u> of the digraph $\Gamma(A,E)$ and is denoted by $e(i,j)$; $e(i,j)$ is said to be an edge from P_i to P_j and may be represented by a directed line from P_i to P_j.

In the digraph $\Gamma(A,E)$, $e(i,j) \in E$ is also said to be an edge from set $\underline{a}_r \subset A$ to set $\underline{a}_s \subset A$, if $P_i \in \underline{a}_r$ and $P_j \in \underline{a}_s$.

DEFINITION 1.1.4. A <u>dipath</u> (directed path) of length L from P_i to P_j is an ordered $(L + 1)$-tuple of points of $\Gamma(A,E)$,

$$\left\langle P_i, P_{k_1}, P_{k_2}, P_{k_3}, \ldots, P_{k_{(L-1)}}, P_j \right\rangle ,$$

in which L is a positive integer and

$$\{ e(i,k_1), e(k_1,k_2), e(k_2,k_3), \ldots , e(k_{(L-1)},j) \}$$

is a subset of the edge set E of $\Gamma(A,E)$. The point P_i is called the <u>initial point</u>, the points $P_{k_1}, P_{k_2}, P_{k_3}, \ldots ,$ $P_{k_{(L-1)}}$ are called <u>intermediate points</u>, and P_j is called the <u>terminal point</u> of the dipath.

We note that a dipath is always of positive finite length and that the dipath $\left\langle P_i, P_j \right\rangle$ of length one is simply the edge $e(i,j)$.

DEFINITION 1.1.5. An edge from P_i to P_i is called a <u>loop</u> at P_i and is denoted by $e(i,i)$.

DEFINITION 1.1.6. If there exists a dipath from P_i to P_j in $\Gamma(A,E)$, we say that P_i <u>is accessible to</u> P_j (or that P_j is accessible from P_i) and denote this by $\mathcal{A}(i,j)$. In this situation, the ordered pair (P_i,P_j) is called an accessible pair.

If P_i <u>is not accessible to</u> P_j, we write $\tilde{\mathcal{A}}(i,j)$.

In this ~~thesis~~ report, <u>we adopt the convention that P_i is always a member of the set of points accessible from P_i; we denote this fact by $\mathcal{A}(i,i)$.</u> It should be noted that $\mathcal{A}(i,i)$ does not necessarily imply that there exists a dipath from P_i to P_i, but does imply that $P_i \in \{P_j : \mathcal{A}(i,j)\}$ for each $P_i \in A$.

DEFINITION 1.1.7. If both $\mathcal{A}(i,j)$ and $\mathcal{A}(j,i)$, that is if P_i is accessible to P_j and P_j is accessible to P_i, we say that P_i and P_j are <u>symmetrically accessible</u> and denote this by $\mathcal{A}^*(i,j)$.

We note that the relation $\mathcal{A}^*$ is an equivalence relation on set A and thus partitions set A (see Bhargava [3]).

DEFINITION 1.1.8. $\Gamma(A,E)$ is a <u>transitive digraph</u> if $e(i,j) \in E$ and $e(j,k) \in E$ implies that $e(i,k) \in E$.

DEFINITION 1.1.9. An <u>edge function</u> f on the digraph $\Gamma(A,E)$ is a function which assigns a non-negative real number to each element of $A \times A$, such that $f(i,j) > 0$ iff $e(i,j) \in E$. $f(i,j) = 0$ otherwise.

DEFINITION 1.1.10 The <u>characteristic edge function</u> f_c on the digraph $\Gamma(A,E)$ is defined to be:

$$f_c(i,j) = \begin{cases} 1, & \text{if } e(i,j) \in E \\ 0, & \text{if } e(i,j) \notin E. \end{cases}$$

In general, the topological definitions and notations used in this report are quite standard and may be found in Kelley [5].

SECTION 1.2. CLASSIFICATION SYSTEMS

There are clearly many ways of describing classification systems for digraphs. We present below two of these which are relevant to our discussion. These are given in Bhargava [3], pp. 7.

CONNECTEDNESS CLASSIFICATION

A digraph $\Gamma(A,E)$ is said to be:

(i) <u>strongly connected</u>, if $\mathcal{Q}^*(i,j)$ for every P_i and P_j in A.

(ii) <u>unilaterally connected</u>, if $\mathcal{Q}(i,j)$ or $\mathcal{Q}(j,i)$, for every P_i and P_j in A.

(iii) <u>weakly connected</u>, if $\Gamma(A,E \cup E')$ is strongly connected, where $E' = \{e(j,i) : e(i,j) \in E\}$.

(iv) <u>disconnected</u>, if $\Gamma(A,E)$ is not even weakly connected.

To obtain a mutually exclusive, totally exhaustive classification system, we may define a digraph $\Gamma(A,E)$ to be of type:

(i) s_3, if $\Gamma(A,E)$ is strongly connected.

(ii) s_2, if $\Gamma(A,E)$ is unilaterally, but not strongly connected.

(iii) s_1, if $\Gamma(A,E)$ is weakly, but not unilaterally connected.

(iv) s_0, if $\Gamma(A,E)$ is disconnected.

A digraph of type s_i is said to be in the connectedness state s_i, for $i = 0, 1, 2,$ or 3.

ACCESSIBILITY CLASSIFICATION

A digraph $\Gamma(A,E)$ is of type s'_k, if the set $\{(P_i,P_j)$: $(P_i,P_j) \in A \times A, \mathcal{Q}(i,j)\}$ contains exactly k members.

CHAPTER 2.[1]

POINT-SET TOPOLOGY AND DIGRAPHS

In this chapter we investigate the accessibility of points of a digraph (see section 1.1.) and the connectedness state of a digraph (see section 1.2.) in terms of some of the concepts of point-set topology.

SECTION 2.1. A TOPOLOGY ON A DIGRAPH

A topology may be determined on set A by suitably defining certain subsets of A to be open with respect to a digraph $\Gamma(A,E)$.

DEFINITION 2.1.1. Set $\underline{a} \subseteq A$ of digraph $\Gamma(A,E)$ is <u>open</u>, if $P_i \in (A \sim \underline{a})$ and $P_j \in \underline{a}$ implies that $e(i,j) \notin E$. In other words, $\underline{a} \subseteq A$ is open if there does not exist an edge in $\Gamma(A,E)$ from $(A \sim \underline{a})$ to $\underline{a}$.

We mention, in passing, that set $\underline{a} \subseteq A$ of $\Gamma(A,E)$ is <u>closed</u> iff set $(A \sim \underline{a})$ of $\Gamma(A,E)$ is <u>open</u>. Hence set $\underline{a} \subseteq A$ of digraph $\Gamma(A,E)$ is closed iff $P_i \in \underline{a}$ and $P_j \in (A \sim \underline{a})$ implies that $e(i,j) \notin E$. That is, set $\underline{a} \subseteq A$ is closed iff there does not exist an edge in $\Gamma(A,E)$ from $\underline{a}$ to $(A \sim \underline{a})$.

---

[1] The main results of this chapter have been presented at the April, 1964 meetings of the American Mathematical Society; see [4].

THEOREM 2.1.1. Each digraph $\Gamma(A,E)$ determines a unique topological space $(A,\mathcal{T})$, where $\mathcal{T} = \{\underline{a}: \underline{a} \subseteq A$ of $\Gamma(A,E)$ is open$\}$. Moreover, $(A,\mathcal{T})$ has completely additive closure (i.e. the intersection of any number of open sets is open).

PROOF. Let $\Gamma(A,E)$ be an arbitrary digraph. Relative to definition 2.1.1., $\Gamma(A,E)$ determines a unique family $\mathcal{T}$, where $\mathcal{T} = \{\underline{a}: \underline{a} \subseteq A$ of $\Gamma(A,E)$ is open$\}$.

To prove that $(A,\mathcal{T})$ is a topological space, we must show that (see Kelley [5], pp. 37): (i) the union of the members of the family $\mathcal{T}$ is A, (ii) the union of the members of each subfamily of $\mathcal{T}$ is a member of $\mathcal{T}$, and (iii) the intersection of any two members of $\mathcal{T}$ is a member of $\mathcal{T}$.

(i) $(A \sim A) = \emptyset$, the void set. Thus, set A vacuously satisfies definition 2.1.1. and thus $A \in \mathcal{T}$. For every $\underline{a} \in \mathcal{T}$, $\underline{a} \subseteq A$. Therefore, $\cup\{\underline{a}: \underline{a} \in \mathcal{T}\} = A$.

(ii) Let $\{\underline{a}_k: k \in K\}$ be an arbitrary subfamily of $\mathcal{T}$. For each $k \in K$, there does not exist an edge from set $(A \sim \underline{a}_k)$ to set $\underline{a}_k$. Thus there does not exist an edge from $\cap\{A \sim \underline{a}_k: k \in K\}$ to any $\underline{a}_k$. Hence there does not exist an edge from $A \sim \cup\{\underline{a}_k: k \in K\}$ to $\cup\{\underline{a}_k: k \in K\}$, i.e. $\cup\{\underline{a}_k: k \in K\} \in \mathcal{T}$.

(iii) Again let $\{\underline{a}_k: k \in K\}$ be an arbitrary subfamily of $\mathcal{T}$. For each $k \in K$, there does not exist an edge from set $(A \sim \underline{a}_k)$ to set $\underline{a}_k$. Thus there does not exist an edge from any $(A \sim \underline{a}_k)$ to $\cap\{\underline{a}_k: k \in K\}$. Hence there does not exist an edge from $\cup\{A \sim \underline{a}_k: k \in K\} =$

$A \sim \cap \{\underline{a}_k : k \in K\}$ to $\cap \{\underline{a}_k : k \in K\}$, i.e. $\cap \{\underline{a}_k : k \in K\} \in \mathcal{T}$.

Consequently, $(A, \mathcal{T})$ is a topolgical space and thus $\tau(A,E)$ determines a unique topological space $(A, \mathcal{T})$. Moreover, by part (iii), (A, τ) has completely additivie **closure**.

LEMMA 2.1.2. Let P_i and P_j be fixed points of set A. $\mathcal{Q}(i,j)$, i.e. P_i is accessible to P_j, iff for each subset $\underline{a} \subseteq A$ containing P_i but not P_j, there exists an edge from $\underline{a}$ to $(A \sim \underline{a})$, i.e. an edge from some point of $\underline{a}$ to some point of $(A \sim \underline{a})$.

PROOF. If $P_i = P_j$, the lemma is trivially true.

Assume that $\mathcal{Q}(i,j)$ for $P_i \neq P_j$. Thus there exists a dipath of finite length from P_i to P_j. Let $\underline{a}$ be an arbitrary subset of A containing P_i but not P_j. In definition 1.1.4.,we note that a dipath is an ordered tuple of finite length. Let P_k, the k th point of the tuple, be the first point of this dipath which is not in set $\underline{a}$, i.e. $P_k \notin \underline{a}$ and each point preceding P_k in the tuple is in $\underline{a}$. Also $P_k \neq P_i$. Thus $P_{(k-1)} \in \underline{a}$, and we have the required edge, namely $e(k-1,k)$.

Let P_i and P_j be distinct fixed points of set A. Assume that for each subset $\underline{a}$ containing P_i but not P_j, there exists an edge from set $\underline{a}$ to set $(A \sim \underline{a})$. Form the set $\underline{a}_i = \{P_h : \mathcal{Q}(i,h)\}$, the set of all points to which P_i is accessible. Assume that $P_j \notin \underline{a}_i$. Then, by hypothesis, there exists an edge from $\underline{a}_i$ to $(A \sim \underline{a}_i)$, say $e(r,s)$ from

$P_r \in \underline{a}_i$ to $P_s \in (A \sim \underline{a}_i)$. But P_i is accessible to P_r: i.e. $P_i = P_r$ or there exists a finite dipath from P_i to P_r. Thus there exists a finite dipath from P_i to P_s which includes the point P_r. Thus P_i is accessible to P_s and therefore $P_s \in \underline{a}_i$. Contradiction! Thus $P_j \in \underline{a}_i$, i.e. P_i is accessible to P_j.

THEOREM 2.1.3. Let P_i and P_j be fixed points of set A. $\mathcal{Q}(i,j)$, i.e. P_i is accessible to P_j, iff each closed set containing P_i contains P_j; or equivalently, iff each open set containing P_j contains P_i.

PROOF. Let P_i and P_j be fixed points of set A.

Assume that $\mathcal{Q}(i,j)$. Let $\underline{a}$ be an arbitrary closed set containing P_i. If $P_j \in (A \sim \underline{a})$, then, by lemma 2.1.2., there exists an edge from $\underline{a}$ to $(A \sim \underline{a})$, i.e. $\underline{a}$ is not closed. Thus $P_j \in \underline{a}$.

Now assume that each closed set containing P_i contains P_j, i.e. there does not exist a closed set containing P_i but not P_j. A set $\underline{a}$ is closed iff the set $(A \sim \underline{a})$ is open; thus there does not exist an open set containing P_j but not P_i, i.e. each open set containing P_j contains P_i.

Finally assume that each open set containing P_j contains P_i. Hence each set $\underline{a}$ containing P_j but not P_i is not open, i.e. each set $(A \sim \underline{a})$ is not closed. Thus for each set $(A \sim \underline{a})$, there exists an edge from $(A \sim \underline{a})$ to $\underline{a}$. By lemma 2.1.2., P_i is accessible to P_j.

SECTION 2.2. CONNECTEDNESS STATES

We present now an identification theorem for the connectedness states (section 1.2.) of a digraph $\Gamma(A,E)$ in terms of the topology $(A,\mathcal{T})$ on that digraph.

THEOREM 2.2.1.

(i) The digraph $\Gamma(A,E)$ is <u>strongly connected</u> (s_3) iff $(A,\mathcal{T})$ is an indiscrete topological space, (i.e. $\mathcal{T} = \{A,\phi\}$).

(ii) The digraph $\Gamma(A,E)$ is <u>unilaterally connected</u> (s_2 or s_3) iff the family of open sets, $\mathcal{T}$, is linearly ordered by inclusion, (i.e. whenever $\underline{a}_1$ and $\underline{a}_2$ are open, then $\underline{a}_1 \subset \underline{a}_2$ or $\underline{a}_2 \subset \underline{a}_1$).

(iii) The digraph $\Gamma(A,E)$ is <u>weakly connected</u> (s_1 or s_2 or s_3) iff $(A,\mathcal{T})$ is "topologically connected", (i.e. set A cannot be expressed as the union of two disjoint non-void open sets).

(iv) The digraph $\Gamma(A,E)$ is <u>disconnected</u> (s_0) iff $(A,\mathcal{T})$ is not "topologically connected", (i.e. set A can be expressed as the union of two disjoint non-void open sets).

PROOF.

(i) Assume that $\Gamma(A,E)$ is strongly connected (see section 1.2.). If P_i and P_j are arbitrary points of $\Gamma(A,E)$, then

$\mathcal{Q}(i,j)$, i.e. P_i is accessible to P_j. By theorem 2.1.3., each open set containing P_j contains P_i. Hence the only non-void open set in $(A,\mathcal{T})$ is A. Thus, $\mathcal{T} = \{A,\phi\}$.

Assume now that $\mathcal{T} = \{A,\phi\}$. Let P_i and P_j be arbitrary points of $\Gamma(A,E)$. Each open set containing P_i contains P_j and each open set containing P_j contains P_i, since A is the only non-void open set in $(A,\mathcal{T})$. By theorem 2.1.3., $\mathcal{Q}(j,i)$ and $\mathcal{Q}(i,j)$, i.e. $\mathcal{Q}^*(i,j)$. Consequently $\Gamma(A,E)$ is strongly connected.

(ii) The family $\mathcal{T}$ of open sets of $(A,\mathcal{T})$ is linearly ordered by inclusion (i.e. for every two open sets of $(A,\mathcal{T})$, one is a subset of the other) iff for arbitrary points P_i and P_j of A there does not exist an open set containing P_i but not P_j or there does not exist am open set containing P_j but not P_i. That is, each open set containing P_j contains P_i or each open set containing P_i contains P_j. This is true iff, by theorem 2.1.3., $\mathcal{Q}(i,j)$ or $\mathcal{Q}(j,i)$, for the arbitrary points P_i and P_j of $\Gamma(A,E)$. By section 1.2., $\Gamma(A,E)$ is unilaterally connected.

(iii) In the digraph $\Gamma(A,E)$, set A cannot be expeessed as the union of two disjoint non-void open sets iff every non-void proper subset of A is not open or is not closed. Equivalently, by definition 2.1.1., for each non-void proper subset, say $\underline{a}_k$, of A, there exists an edge from $(A \sim \underline{a}_k)$ to $\underline{a}_k$ or there exists an edge from $\underline{a}_k$ to $(A \sim \underline{a}_k)$ in

$\Gamma(A,E)$. That is, in $\Gamma(A, E \cup E')$, where $E' = \{e(j,i) : e(i,j) \in E\}$, there exists an edge from $(A \sim \underline{a}_k)$ to $\underline{a}_k$ and there exists an edge from $\underline{a}_k$ to $(A \sim \underline{a}_k)$, for each non-void proper subset $\underline{a}_k$ of A. Hence by definition 2.1.1., the only open sets in $\Gamma(A, E \cup E')$ are ϕ and A. This is true iff, by part (i) of this theorem, $\Gamma(A, E \cup E')$ is strongly connected. Equivalently, by section 1.2, $\Gamma(A,E)$ is weakly connected.

(iv) This is the contrapositive of part (iii) of this theorem.

In section 2.4., we discuss the closure operator and present an identification theorem (thm. 2.4.4.) based upon the closure operator. It is equivalent to the theorem that we have just proved.

We note that the connectedness classification of a digraph $\Gamma(A,E)$ is consistent with the "topological connectedness" of the topological space $(A,\mathfrak{J})$ determined by that digraph. The topological space determined by a digraph is dependent upon the definition of an "open set". In chapter 3, we show that alternate definitions for an "open set" may or may not produce this consistency. Throughout this report, an open set refers to a set satisfying definition 2.1.1.

SECTION 2.3. SEPARATION AXIOMS

In this section, we investigate a digraph in terms of the separation axiom(s) satisfied by the topological space which is determined by that digraph. We make use of the following two separation axioms (see Kelley [5], pp. 56-57): (i) a topological space <u>$(A,\mathcal{T})$ is a T_0-space</u> iff for every $P_i \neq P_j$, either there exists an open set containing P_i but not P_j or there exists an open set containing P_j but not P_i; and (ii) a topological space <u>$(A,\mathcal{T})$ is a T_1-space</u> iff each set which consists of a single point is closed.

THEOREM 2.3.1.

(i) In digraph $\Gamma(A,E)$, $\tilde{\mathcal{Q}}(i,j)$ or $\tilde{\mathcal{Q}}(j,i)$, for all $P_i \neq P_j$ (i.e. there does not exist P_i and P_j such that $P_i \neq P_j$ and $\mathcal{Q}^*(i,j)$) iff $(A,\mathcal{T})$ is a <u>T_0-space</u>.

(ii) In digraph $\Gamma(A,E)$, $\{e(i,j) : e(i,j) \in E,\ i \neq j\} = \emptyset$ iff $(A,\mathcal{T})$ is a <u>T_1-space</u>; or equivalently, iff $(A,\mathcal{T})$ is a discrete space.

PROOF.

(i) Let P_i add P_j denote distinct arbitrary points of $\Gamma(A,E)$. $\tilde{\mathcal{Q}}(i,j)$ or $\tilde{\mathcal{Q}}(j,i)$, i.e. P_i is not accessible to P_j or P_j is not accessible to P_i. This is true iff, by theorem 2.1.3., there exists an open set containing P_j but not P_i or there exists an open set containing P_i but not P_j, i.e. $(A,\mathcal{T})$ is a T_0-space.

(ii) For digraph $\Gamma(A,E)$, $\{e(i,j) : e(i,j) \in E, i \neq j\} = \phi$, i.e. there does not exist an edge from any point of A to any other distinct point of A, iff (using definition 2.1.1.) each set consisting of a single point is closed, i.e. $(A,\mathcal{T})$ is a T_1-space. Since $(A,\mathcal{T})$ has completely additive closure, each set consisting of a single point is closed iff every subset of A is open, i.e. $(A,\mathcal{T})$ is a discrete space.

SECTION 2.4. TOPOLOGICAL OPERATORS

In this section, we make use of the closure, kernel, and core operators to investigate the connectedness of a digraph. The standard topological definition is given for the closure of a set. The definitions for the kernel and the core of a set are taken from an article by Aull and Thron [1].

First let us state the definition of the closure of a set in the topological space $(A,\mathcal{T})$.

DEFINITION 2.4.1. The closure of set $\underline{a}$, denoted by $Cl(\underline{a})$, is the intersection of all the closed subsets of A containing $\underline{a}$, i.e. $Cl(\underline{a}) = \cap\{\underline{a}_j : \underline{a}_j \text{ is closed}, \underline{a} \subseteq \underline{a}_j \subseteq A\}$.

Let us now define an operator analogous to the closure operator in terms of the open sets rather than the closed sets containing a particular set.

DEFINITION 2.4.2. The <u>kernel</u> of set $\underline{a}$, denoted by $Kl(\underline{a})$, is the intersection of all the open subsets of A containing set $\underline{a}$, i.e. $Kl(\underline{a}) = \cap \{\underline{a}_j : \underline{a}_j$ is open, $\underline{a} \subseteq \underline{a}_j \subseteq A\}$.

We note that the closure of a set is always closed and that set $\underline{a}$ is closed iff $\underline{a} = Cl(\underline{a})$. Also, since the topology on a digraph has completely additive closure (see theorem 2.1.1.), we see that the kernel of a set is open and that set $\underline{a}$ is open iff $\underline{a} = Kl(\underline{a})$.

THEOREM 2.4.1.

(i) For any point P_i of $\Gamma(A,E)$,

$$Cl\ \{P_i\} = \{P_j : \mathcal{Q}(i,j)\}, \text{ and}$$
$$Kl\ \{P_i\} = \{P_j : \mathcal{Q}(j,i)\}.$$

In words, $Cl\ \{P_i\}$ is the set of points of the digraph which are accessible from P_i. $Kl\ \{P_i\}$ is the set of points of the digraph which are accessible to point P_i.

(ii) For any set $\underline{a} \subseteq A$ of $\Gamma(A,E)$,

$$Cl(\underline{a}) = \{P_j : \mathcal{Q}(i,j) \text{ for some } P_i \in \underline{a}\}, \text{ and}$$
$$Kl(\underline{a}) = \{P_j : \mathcal{Q}(j,i) \text{ for some } P_i \in \underline{a}\}.$$

PROOF.

(i) By definition 2.4.1., $Cl\ \{P_i\} = \cap \{\underline{a}_j : \underline{a}_j$ is closed and $P_i \in \underline{a}_j\}$. That is, $Cl\ \{P_i\}$ is the set of all points such that every closed set containing P_i contains that point. Therefore, applying theorem 2.1.3., we have $\mathrm{Cl}\ \{P_i\} = \{P_j : \mathcal{Q}(i,j)\}$. Similarly, from definition 2.4.2. and theorem 2.1.3., we have that $\mathrm{Kl}\ \{P_i\} = \{P_j : \mathcal{Q}(j,i)\}$.

(ii) The topology $(A,\mathcal{T})$ on a digraph $(A,\mathcal{T})$ has completely additive closure; therefore, $\mathrm{Cl}\ (\cup \{\underline{a}_i : i \in I\}) = \cup \{\mathrm{Cl}\ \{\underline{a}_i\} : i \in I\}$ and $\mathrm{Kl}\ (\cup \{\underline{a}_j : j \in J\}) = \cup \{\mathrm{Kl}\ \{\underline{a}_j\} : j \in J\}$. Using this fact, along with theorem 2.4.1., we obtain the following.

$$
\begin{aligned}
\mathrm{Cl}(\underline{a}) &= \mathrm{Cl}\ (\ \cup \{\ \{P_i\} : P_i \in \underline{a}\}\ ) \\
&= \cup \{\ \mathrm{Cl}\ \{P_i\} : P_i \in \underline{a}\ \} \\
&= \cup \{\ \{P_j : \mathcal{Q}(i,j)\} : P_i \in \underline{a}\ \} \\
&= \{\ P_j : \mathcal{Q}(i,j) \text{ for some } P_i \in \underline{a}\ \}. \\
\mathrm{Kl}(a) &= \mathrm{Kl}\ (\ \cup \{\ \{P_i\} : P_i \in \underline{a}\ \}\ ) \\
&= \cup \{\ \mathrm{Kl}\ \{P_i\} : P_i \in \underline{a}\ \} \\
&= \cup \{\ \{P_j : \mathcal{Q}(j,i)\} : P_i \in \underline{a}\ \} \\
&= \{\ P_j : \mathcal{Q}(j,i) \text{ for some } P_j \in \underline{a}.
\end{aligned}
$$

We wish to note here that Berge defines the "transitive closure" of a point P_i to be a set of points identical with, in our terminology, the set $\{P_k : \mathcal{Q}(i,k)\}$. Thus, from the results of theorem 2.4.1., it follows that, in a digraph, our closure of P_i is equivalent to Berge's "transitive closure" of P_i. (Berge [2], pp. 11).

The closure operator is used to define various topological terms, e.g.: a set $\underline{a}$ is <u>dense</u> in a topological space $(A,\mathcal{T})$ if $\mathrm{Cl}(\underline{a}) = A$ (see Kelley [5], pp. 49). Two subsets

$\underline{a}_i$ and $\underline{a}_j$ are separated in $(A,\mathcal{T})$ if $Cl(\underline{a}_i) \cap \underline{a}_j = \phi$ and $\underline{a}_i \cap Cl\ \underline{a}_j = \phi$ (see Kelley [5], pp. 52). Let us relate dense and separated sets to the accessibility of points in a digraph.

COROLLARY 2.4.2. Set $\underline{a}$ is dense in A of $\Gamma(A,E)$ iff for each point P_k of A, there exists a point P_i of $\underline{a}$ such that P_i is accessible to P_k.

PROOF. Follows from theorem 2.4.1.

COROLLARY 2.4.3. Two subsets $\underline{a}_i$ and $\underline{a}_j$ of A are separated in $\Gamma(A,E)$ iff there does not exist $P_i \in \underline{a}_i$ and $P_j \in \underline{a}_j$, such that $\mathcal{A}(i,j)$ or $\mathcal{A}(j,i)$.

PROOF. $Cl(\underline{a}_i) \cap \underline{a}_j = \phi$ and $\underline{a}_i \cap Cl\ \underline{a}_j = \phi$ iff, by theorem 2.4.1., there does not exist points $P_i \in \underline{a}_i$ and $P_j \in \underline{a}_j$ such that $\mathcal{A}(i,j)$ and there does not exist points $P_r \in \underline{a}_i$ and $P_s \in \underline{a}_j$ such that $\mathcal{A}(s,r)$.

In section 2.2, we proved an identification theorem for the connectedness states of a digraph. We now present an identification theorem based upon the closure operator and separated sets.

THEOREM 2.4.4.

(i) The digraph $\Gamma(A,E)$ is s_0 (disconnected) iff set A can be expressed as the union of two non-void separated subsets of $(A,\mathcal{T})$.

(ii) The digraph $\Gamma(A,E)$ is s_0 (disconnected) or s_1 (weakly connected, but not unilaterally connected) iff $(A,\mathcal{T})$ contains two non-void separated subsets.

(iii) The digraph $\Gamma(A,E)$ is s_3 (strongly connected) iff each point of A is dense in $(A,\mathcal{T})$, i.e. iff for each $P_i \in A$, $Cl\,\{P_i\} = A$.

PROOF.

(i) $\Gamma(A,E)$ is s_0 iff, by theorem 2.2.1., set A can be expressed as the union of two disjoint non-void open sets, say $\underline{a}_i$ and $(A \sim \underline{a}_i)$. That is, there does not exist an edge from set $\underline{a}_i$ to $(A \sim \underline{a}_i)$ or from set $(A \sim \underline{a}_i)$ to $\underline{a}_i$. Hence, $Cl(\underline{a}_i) = \underline{a}_i$ and $Cl(A \sim \underline{a}_i) = (A \sim \underline{a}_i)$. Therefore, $\Gamma(A,E)$ is s_0 iff sets $\underline{a}_i$ and $(A \sim \underline{a}_i)$ are separated in $(A,\mathcal{T})$.

(ii) Assume that $\Gamma(A,E)$ is either s_1 or s_0, i.e. $\Gamma(A,E)$ is neither s_3 nor s_2. Then, by section 1.2., there exist points, say P_i and P_j, in A such that $\tilde{\mathcal{Q}}(i,j)$ and $\tilde{\mathcal{Q}}(j,i)$. By theorem 2.4.1., $P_i \notin Cl\,\{P_j\}$ and $P_j \notin Cl\,\{P_i\}$. Thus, $\{P_i\}$ and $\{P_j\}$ are two non-void separated subsets of $(A,\mathcal{T})$.

Assume now that $(A,\mathcal{T})$ contains two non-void separated subsets, say, $\underline{a}_i$ and $\underline{a}_j$. By corollary 2.4.3., there does not exist $P_i \in \underline{a}_i$ and $P_j \in \underline{a}_j$, such that $\mathcal{Q}(i,j)$ or $\mathcal{Q}(j,i)$, since $\underline{a}_i$ and $\underline{a}_j$ are non-void, there exist points, say $P_r \in \underline{a}_i$ and $P_s \in \underline{a}_j$, such that $\tilde{\mathcal{Q}}(r,s)$ and $\tilde{\mathcal{Q}}(s,r)$.

By section 1.2., $\Gamma(A,E)$ is either s_0 or s_1.

(iii) By theorem 2.2.1., $\Gamma(A,E)$ is strongly connected iff $\tau = \{A, \phi\}$. This is true iff $Cl\{P_i\} = A$ for each $P_i \in A$.

Let us now define the core operator and relate it to the closure and kernel operators.

DEFINITION 2.4.3. The <u>core</u> of set $\underline{a}$, denoted by $\mathcal{K}(\underline{a})$, is the intersection of all subsets of A containing $\underline{a}$ which are closed or open, i.e.

$\mathcal{K}(\underline{a}) = \cap \{\underline{a}_j : \underline{a}_j$ is closed or open, $\underline{a} \subseteq \underline{a}_j \subseteq A\}$. We note that $\mathcal{K}(\underline{a}) = Cl(\underline{a}) \cap Kl(\underline{a})$.

THEOREM 2.4.5.

(i) For any point P_i of $\Gamma(A,E)$,

$$\mathcal{K}\{P_i\} = \{P_j : \mathcal{Q}^*(i,j)\}.$$

In words, $\mathcal{K}\{P_i\}$ is the set of all points of the digraph which are symmetrically accessible to P_i.

(ii) For any set $\underline{a} \subseteq A$ of $\Gamma(A,E)$, $\mathcal{K}(\underline{a}) = \{P_j : \mathcal{Q}(i,j)$ for some $P_i \in A$ and $\mathcal{Q}(j,k)$ for some $P_k \in \underline{a}\}$.

PROOF. Using theorem 2.4.1. and definition 2.4.3., we obtain the following.

(i) $\mathcal{K}\{P_i\} = Cl\{P_i\} \cap Kl\{P_i\}$

$$= \{P_j : \mathcal{Q}(i,j)\} \cap \{P_j : \mathcal{Q}(j,i)\}$$

$$= \{P_j : \mathcal{Q}(i,j) \text{ and } \mathcal{Q}(j,i)\} = \{P_j : \mathcal{Q}^*(i,j)\}$$

(ii) $\mathcal{K}(\underline{a}) = Cl(\underline{a}) \cap Kl(\underline{a})$

$$= \{P_j : \mathcal{Q}(i,j) \text{ for some } P_i \in \underline{a}\}$$
$$\cap \{P_j : \mathcal{Q}(j,k) \text{ for some } P_k \in \underline{a}\}$$
$$= \{P_j : \mathcal{Q}(i,j) \text{ for some } P_i \in \underline{a} \text{ and}$$
$$\mathcal{Q}(j,k) \text{ for some } P_k \in \underline{a}\}.$$

We note that $\mathcal{K}(\cup \{\underline{a}_i : i \in I\}) \supseteq \cup \{\mathcal{K}(\underline{a}_i) : i \in I\}$ and thus for any $\underline{a} \subseteq A$, $\mathcal{K}(\underline{a}) \supseteq \{P_j : \mathcal{Q}^*(i,j)$ for some $P_i \in \underline{a}\}$.

COROLLARY 2.4.6. For any point P_i of the digraph $\Gamma(A,E)$, the core of P_i is the maximum subset, say $\underline{a}$, of A containing P_i such that the subgraph $\Gamma(\underline{a})$ is strongly connected (s_3).

PROOF. By theorem 2.4.5., $P_k \in \mathcal{K}\{P_i\}$ iff $\mathcal{Q}^*(i,k)$. Thus by the connectedness classification (section 1.2.), the subdigraph $\Gamma(\mathcal{K}\{P_i\})$ is strongly connected. For any set $\underline{a}_i$ containing P_i and containing some other point, say P_j, where $P_j \notin \mathcal{K}\{P_i\}$, $\Gamma(\underline{a}_i)$ is not strongly connected.

SECTION 2.5. TOPOLOGIES AND TRANSITIVE DIGRAPHS

In this section, we establish a one-to-one mapping of a certain class of digraphs onto a certain class of topologies.

Let A be a fixed, but arbitrary, set of points and let $\mathcal{E} = \{E : \phi \subseteq E \subseteq A \times A\}$, i.e. $\mathcal{E}$ is the family of all subsets of the cartesian product $A \times A$.

From theorem 2.1.1., we see that each digraph $\Gamma(A,E)$ determines a unique topological space $(A,\mathcal{T})$ and this topological space has completely additive closure. This establishes a mapping of the family of all digraphs, $\mathcal{W} = \{\Gamma(A,E) : E \in \mathcal{E}\}$, on set A into the family of topologies, $\mathcal{Y} = \{ (A,\tau) : (A,\mathcal{T})$ has completely additive closure $\}$, on set A. Let us demote this mapping by φ.

PROPOSITION 2.5.1.

(i) For any digraph $\Gamma(A,E)$, $\varphi(\Gamma(A,E)) =$ (the topology (A,τ) which has the family $\{ \{P_i : \mathcal{Q}(i,j) \text{ in } \Gamma(A,E)\} : P_j \in A \} = \{Kl\ \{P_j\} : P_j \in A\}$ as a base).

(ii) φ is a many-to-one mapping of the family $\mathcal{W}$ of all digraphs on set A onto the family $\mathcal{Y}$ of all topologies with completely additive closure on set A.

PROOF.

(i) $\varphi(\Gamma(A,E)) = (A,\mathcal{T})$ is the topology determined by the digraph $\Gamma(A,E)$ relative to definition 2.1.1. of an open set. For each point P_j of $\Gamma(A,E)$, the set $Kl\ \{P_j\}$ is open and is the smallest open set containing the point P_j. Thus $\{Kl\ \{P_j\} : P_j \in A\}$ is a base for the topology $(A,\mathcal{T})$.

By theorem 2.4.1., for each P_j in $\Gamma(A,E)$, $Kl\ \{P_j\} = \{P_i : \mathcal{Q}(i,j) \text{ in } \Gamma(A,E)\}$.

(ii) φ is a many-to-one mapping, by theorem 2.1.1. If $(A,\mathcal{T})$ is an arbitrary topology on set A and if

$E = \{e(i,j) : P_i \in Kl\{P_j\}$ in $(A,\tau)\}$, then $\Gamma(A,E)$ is one of the members of $\mathcal{W}$, such that $\varphi(\Gamma(A,E)) = (A,\tau)$. Hence φ is an onto mapping.

We wish now to determine a subfamily of $\mathcal{W}$ such that the mapping φ restricted to this subfamily is a one-to-one mapping onto the family $\mathcal{Y}$.

Let $\mathcal{U} = \{\Gamma(A,E) : e(i,j) \in E$ iff $\mathcal{Q}(i,j)$ in $\Gamma(A,E)\}$.

PROPOSITION 2.5.2. $\mathcal{U}$ is the family of transitive digraphs, with loops, on set A. That is:

$\Gamma(A,E) \in \mathcal{U}$ if and only if

(i) $e(i,j) \in E$ and $e(j,k) \in E$ implies that $e(i,k) \in E$,

and,

(ii) $\{e(i,i) : P_i \in A\} \subseteq E$.

PROOF. The relation of accessibility is transitive and reflexive. $\mathcal{U} = \{\Gamma(A,E) : e(i,j) \in E$ iff $\mathcal{Q}(i,j)$ in $\Gamma(A,E)\}$. Thus $\Gamma(A,E) \in \mathcal{U}$ iff $\Gamma(A,E)$ is a transitive digraph (see definition 1.1.8.) and $\Gamma(A,E)$ has a loop (see defintion 1.1.5.) at each point of set A.

Let us consider the mapping φ restricted to the family $\mathcal{U} \subseteq \mathcal{W}$, i.e. the mapping $(\varphi \mid \mathcal{U})$.

PROPOSITION 2.5.3. Let A be a fixed set of points.

(i) $(\varphi|\mathcal{U})$ $(\Gamma(A,E))$ = (the topology $(A,\mathcal{T})$ which has the family $\{\{P_i : e(i,j) \in E\} : P_j \in A\} = \{Kl\ \{P_j\} : P_j \in A\}$, as a base).

(ii) $(\varphi|\mathcal{U})$ is a one-to-one mapping of the family $\mathcal{U}$ of all transitive digraph with loops on set A onto the family $\mathcal{Z}$ of all topologies with completely additive closure on set A.

PROOF.

(i) This is proposition 2.5.1. (i) with the mapping φ replaced by $(\varphi|\mathcal{U})$, where $\mathcal{U} = \{\Gamma(A,E) : e(i,j) \in E$ iff $\mathcal{Q}$ (i,j) in $\Gamma(A,E)\}$.

(ii) By part (i), $(\varphi|\mathcal{U})$ is a many-to-one mapping of $\mathcal{U}$ into $\mathcal{Z}$. If $(A,\mathcal{T})$ is an arbitrary member of the family $\mathcal{Z}$ and $E = \{e(i,j) : P_i \in Kl\ \{P_j\}$ in $(A,\mathcal{T})\}$, then $\Gamma(A,E)$ is the unique member of $\mathcal{U}$ such that $(\varphi|\mathcal{U})$ $(\Gamma(A,E)) = (A,\mathcal{T})$. Thus $(\varphi|\mathcal{U})$ is a one-to-one mapping of $\mathcal{U}$ onto $\mathcal{Z}$.

We note that $(\varphi|\mathcal{U})^{-1}$ $(A,\mathcal{T})$ = (the digraph $\Gamma(A,E)$ in which $E = \{e(i,j) : P_i \in Kl\ \{P_j\}$ in $(A,\mathcal{T})\}$). Equivalently $E = \{e(i,j) : P_j \in Cl\ \{P_i\}$ in $(A,\mathcal{T})\}$.

In general, if the set of points is not specified, $(\varphi|\mathcal{U}')$ is a one-to-one mapping of the family $\mathcal{U}'$ of all transitive digraphs with loops onto the family $\mathcal{Z}'$ of all topologies with completely additive closure.

SECTION 2.6. MAXIMUM AND MINIMUM EDGE SETS

By proposition 2.5.1., the mapping φ is a many-to-one mapping of the family of all digraphs on set A onto the family of all topologies with completely additive closure on set A. Thus, distinct digraphs on set A may determine identical topologies on set A. In this section, for an arbitrary topology with completely additive closure, we investigate the existence of a digraph with a maximum edge set and a digraph with a minimum edge set which determine that topology.

DEFINITION 2.6.1.

(I) The digraph $\Gamma(A,E)$ is called the <u>maximum digraph</u> determining the topology $(A,\mathcal{T})$, if

(i) $\varphi(\Gamma(A,E)) = (A,\mathcal{T})$, and

(ii) $E' \subseteq E$, for all E' such that $\varphi(\Gamma(A,E')) = (A,\mathcal{T})$.

(II) The digraph $\Gamma(A,E)$ is called the <u>minimum digraph</u> determining the topology $(A,\mathcal{T})$, if

(i) $\varphi(\Gamma(A,E)) = (A,\mathcal{T})$, and

(ii) $E \subseteq E'$, for all E' such that $\varphi(\Gamma(A,E')) = (A,\mathcal{T})$.

DEFINITION 2.6.2. The digraph $\Gamma(A,E)$ is called a <u>minimal digraph</u> determining the topology $(A,\mathcal{T})$, if

(i) $\varphi(\Gamma(A,E)) = (A,\mathcal{T})$, and

(ii) $\varphi(\Gamma(A,E')) \neq (A,\mathcal{T})$, for all E' such that $E' \subset E$ and $E' \neq E$.

PROPOSITION 2.6.1. Let A be an arbitrary set of points and let $(A,\mathcal{T})$ be an arbitrary topology with completely additive closure on set A. $(\varphi|\mathcal{U})^{-1}\ (A,\mathcal{T}) =$ (the digraph $\Gamma(A,E)$ in which $E = \{e(i,j) : P_j \in \mathrm{Cl}\ \{P_i\}$ in $(A,\mathcal{T})\}$) is the <u>maximum digraph</u> determining $(A,\mathcal{T})$.

PROOF. For any point P_i of $\Gamma(A,E)$, $\mathrm{Cl}\ \{P_i\} = \{P_j : e(i,j) \in E\}$. For any point P_k of any digraph on set A, $\mathrm{Cl}\ \{P_k\} \supseteq \{P_j : e(k,j)$ is an edge of that digraph$\}$. Thus the addition of an edge to set E will necessarily alter the closure of the initial point of that edge and thus alter to topology on set A. Therefore $\Gamma(A,E)$ is the maximum digraph determing the topological space $(A,\mathcal{T})$.

For a topology with completely additive closure there does <u>not</u> necessarily exist a minimum digraph determining that topology. In fact, there does not always exist a minimal digraph.

PROPOSITION 2.6.2. Let A be an arbitrary set of points and let $(A,\mathcal{T})$ be an arbitrary topology with completely additive closure on set A. There does <u>not necessarily exist</u> a <u>minimal digraph</u> determining $(A,\mathcal{T})$.

PROOF. Let set A be the set of all ordinal numbers which are less than or equal to P_ω, the first non-finite ordinal, i.e. $A = \{P_1, P_2, P_3, \ldots P_\omega\}$. Let $\mathcal{T} = \{A\} \cup \{\ \{P_j : P_j < P_k\} : P_k \in A\}$.

Let $\Gamma(A,E)$ be an arbitrary digraph determining the topological space $(A,\mathcal{T})$. The only open set containing P_ω is A. Thus $Kl\ \{P_\omega\} = A$ and, by theorem 2.4.1., $\mathcal{Q}(k,\omega)$ for every $P_k \in A$. That is, for every $P_k \in (A \sim \{P_\omega\})$, there exists a finite dipath from P_k to P_ω. Therefore, there exists in E a countably infinite number of edges from set $(A \sim \{P_\omega\})$ to point P_ω. Let P_l denote the first point of set A such that $e(l,\omega) \in E$. Let $E^\# = (E \sim e(l,\omega))$. Therefore, $\varphi(\Gamma(A,E^\#)) = (A,\mathcal{T})$, where $E^\# \subset E$ and $E^\# \neq E$.

CHAPTER 3.

OTHER TOPOLOGIES ON A DIGRAPH

In this chapter we consider two alternate definitions to the definition of an open set used in Chapter 2. and investigate briefly the topology determined by a digraph with respect to each of these alternate definitions. In all cases, we assume the standard topological concepts, e.g. set $\underline{a} \subseteq A$ is ν-open iff set $(A \sim \underline{a})$ is ν-closed.

SECTION 3.1. OPEN SETS

As convenient reference, we restate several definitions and theorems from Chapter 2.

DEFINITION 2.1.1. Set $\underline{a} \subseteq A$ of digraph $\Gamma(A,E)$ is <u>open</u>, if $P_i \in (A \sim \underline{a})$ and $P_j \in \underline{a}$ implies that $e(i,j) \notin E$. In other words, set $\underline{a} \subseteq A$ is open, if there does not exist an edge in $\Gamma(A,E)$ from set $(A \sim \underline{a})$ to set $\underline{a}$.

THEOREM 2.1.1. Each digraph $\Gamma(A,E)$ determines a unique topological space $(A,\mathcal{T}_E)$, where $\mathcal{T}_E = \{\underline{a} : \underline{a} \subseteq A$ of $\Gamma(A,E)$ is open$\}$. Moreover, $(A,\mathcal{T}_E)$ has completely additive closure.

THEOREM 2.2.1. (iii) The digraph $\Gamma(A,E)$ is <u>weakly connected</u> (s_1 or s_2 or s_3) iff $(A,\mathcal{T}_E)$ is "topologically connected", (i.e. set A cannot be expressed as the union of two disjoint non-void open sets).

SECTION 3.2. γ -OPEN SETS

DEFINITION 3.2.1. Set $\underline{a} \subseteq A$ of digraph $\Gamma(A,E)$ is <u>ν-open</u>, if $P_i \in \underline{a}$ and $P_j \in (A \sim \underline{a})$ implies that $e(i,j) \notin E$. In other words, set $\underline{a} \subseteq A$ is ν-open, if there does not exist an edge in $\Gamma(A,E)$ from set $\underline{a}$ to set $(A \sim \underline{a})$.

PROPOSITION 3.2.1.

(i) Set $\underline{a} \subseteq A$ of digraph $\Gamma(A,E)$ is ν-open iff $\underline{a} \subseteq A$ of $\Gamma(A,E)$ is closed.

(ii) Set $\underline{a} \subseteq A$ of digraph $\Gamma(A,E)$ is ν-open iff set $\underline{a} \subseteq A$ of digraph $\Gamma(A,E')$ is open, where $E' = \{e(j,i) : e(i,j) \in E\}$.

PROOF. Compare definition 3.2.1. with definition 2.1.1.

We note that $(E')' = E$ and that the digraph $\Gamma(A,E')$ may be obtained from the digraph $\Gamma(A,E)$ by reversing the direction of each and every edge of $\Gamma(A,E)$. This operation does not alter the connectedness state (see section 1.2.) of a digraph, thus for $i = 0, 1, 2,$ or 3, $\Gamma(A,E')$ is of type s_i iff $\Gamma(A,E)$ is of type s_i.

LEMMA 3.2.2. Each digraph $\Gamma(A,E)$ determines, with respect to ν-open sets, a unique topological space $(A,\mathcal{T}_E)_\nu$. This topological space is identical to the topological space $(A,\mathcal{T}_{E'})$ determined with respect to open sets by digraph $\Gamma(A,E')$, where $E' = \{e(j,i) : e(i,j) \in E\}$.

PROOF. Let $\underline{a}$ be an arbitrary subset of A. By proposition 3.2.1. (ii), set $\underline{a} \subseteq A$ of $\Gamma(A,E)$ is ν-open iff $\underline{a} \subseteq A$ of $\Gamma(A,E')$ is open. Thus $\mathcal{T}_E = \{\underline{a} : \underline{a} \subseteq A$ of $\Gamma(A,E)$ is ν-open$\}$ is the same family as $\mathcal{T}_{E'} = \{\underline{a} : \underline{a} \subseteq A$ of $\Gamma(A,E')$ is open$\}$. Hence, by theorem 2.1.1., $\Gamma(A,E')$ determines a unique topological space $(A,\mathcal{T}_{E'})$ which is identical to the topological space $(A,\mathcal{T}_E)_\nu$. Therefore, each digraph $\Gamma(A,E)$ determines, with respect to ν-open sets, a unique topological space $(A,\mathcal{T}_E)_\nu$. Moreover, $(A,\mathcal{T}_E)_\nu$ has completely additive closure.

THEOREM 3.2.3. The digraph $\Gamma(A,E)$ is <u>weakly connected</u> (s_1 or s_2 or s_3) iff $(A,\mathcal{T}_E)_\nu$ is "topologically connected" with respect to ν-open sets.

PROOF. $\Gamma(A,E)$ is of the same connectedness state as $\Gamma(A,E')$, where $E' = \{e(j,i) : e(i,j) \in E\}$. In particular, $\Gamma(A,E)$ is weakly connected iff $\Gamma(A,E')$ is weakly connected. By the lemma 3.2.2., $(A,\mathcal{T}_E)_\nu$ is identical to $(A,\mathcal{T}_{E'})$. Substituting into theorem 2.2.1. (iii), we have the following. $\Gamma(A,E)$ is weakly connected iff $(A,\mathcal{T}_E)_\nu$ is "topologically connected" with respect to γ-open sets.

Likewise, parts (i), (ii), and (iv) of theorem 2.2.1. of chapter 2. hold true both when the topology is based upon open sets and when the topology is based upon γ-open sets. Thus the connectedness classification of a digraph $\Gamma(A,E)$ is consistent with the "topological connectedness" of the topological space $(A,\mathcal{T}_E)$ and the topological space $(A,\mathcal{T}_E)_\gamma$.

SECTION 3.3. δ-OPEN SETS

DEFINITION 3.3.1. Set $\underline{a} \subseteq A$ of digraph $\Gamma(A,E)$ is <u>δ-open</u>, if $P_i \in (A \sim \underline{a})$ and $P_j \in \underline{a}$ implies that $e(i,j) \in E$. In other words, set $\underline{a} \subseteq A$ is δ-open, if there exists an edge from each point of set $(A \sim \underline{a})$ to each point of set $\underline{a}$.

PROPOSITION 3.3.1. Set $\underline{a} \subseteq A$ of digraph $\Gamma(A,E)$ is δ-open iff set $\underline{a} \subseteq A$ of digraph $\Gamma(A,E^*)$ is open, where $E^* = (A \times A) \sim E$.

PROOF. Compare definition 3.3.1. with definition 2.1.1.

We note that $(E^*)^* = E$ and that the digraph $\Gamma(A,E^*)$ may be obtained from the digraph $\Gamma(A,E)$ be including in $\Gamma(A,E^*)$ those and only those edges which do not appear in the digraph $\Gamma(A,E)$. This operation does, in some cases, change the connectedness state of a digraph. For example, let $A = \{P_1, P_2\}$. The digraph $\Gamma(A, A \times A)$ is strongly connected (s_3), but the digraph $\Gamma(A, \phi)$ is disconnected (s_0).

LEMMA 3.3.2. Each digraph $\Gamma(A,E)$ determines, with respect to δ-open sets, a unique topological space $(A, \mathcal{T}_E)_\delta$. This topological space is identical to the topological space $(A, \mathcal{T}_{E^*})$ determined with respect to open sets by the digraph $\Gamma(A,E^*)$, where $E^* = (A \times A) \sim E$.

PROOF. Let $\underline{a}$ be an arbitrary subset of A. By proposition 3.3.1., set $\underline{a} \subseteq A$ of $\Gamma(A,E)$ is δ-open iff $\underline{a} \subseteq A$ of $\Gamma(A,E^*)$ is open. Thus $\mathcal{T}_E = \{\underline{a} : \underline{a} \subseteq A$ of $\Gamma(A,E)$ is δ-open$\}$. Is the same family as $\mathcal{T}_{E^*} = \{\underline{a} : \underline{a} \subseteq A$ of $\Gamma(A,E^*)$ is open$\}$. Hence, by theorem 2.1.1., $\Gamma(A,E^*)$ determines a unique topological space (A,E^*) which is identical to the topological space $(A, \mathcal{T}_E)_\delta$. Therefore, each digraph $\Gamma(A,E)$ determines, with respect to δ-open sets, a unique topological space $(A, \mathcal{T}_E)_\delta$. Moreover, $(A, \mathcal{T}_E)_\delta$ has completely additive closure.

In general, the connectedness classification of a digraph $\Gamma(A,E)$ is <u>not</u> consistent with the "topological connectedness" of the topological space $(A, \mathcal{T}_E)_\delta$. For example: let $A = \{P_1, P_2\}$. The digraph $\Gamma(A, \phi)$ is disconnected (s_0), but the topological space $(A, \mathcal{T}_\phi)_\delta$ determined, with respect to δ-open sets, by $\Gamma(A, \phi)$ is an indiscrete space and hence is "topologically connected".

Many other definitions of an open set may be used to establish a topology on a digraph. This author finds definition 2.1.1. of an open set most useful in investigating the accessibility of points and the connectedness classification of a digraph.

Perhaps other approaches may be used to establish a more general topology on a digraph, e.g. a topology which does not necessarily have the property of completely additive closure or a topology which may be T_1 in other than the trivial case. Yet in all cases, we desire a topology whose "topological connectedness" is consistent with the connectedness state of the digraph. One possible approach is to define an open set in terms of an edge function (definition 1.1.9.) assigned to the digraph.

BIBLIOGRAPHY

1. C. E. Aull and W. J. Thron
   Separation axioms between T_0 and T_1, Indagationes Math. 24 (1962), 26-37.

2. C. Berge (translated by A. Doig)
   The Theory of Graphs, Wiley and Sons, New York, 1962.

3. T. N. Bhargava
   A Stochastic Model For Time Changes in a Binary Dyadic Relation, with Application to Group Dynamics, Ph.D. Thesis, Michigan State University, 1962.

4. T. N. Bhargava and T. J. Ahlborn
   Directed graphs and point-set topology, Abstract in Notices Amer. Math. Soc. 11 (1964), 341.

5. J. L. Kelley
   General Topology, Van Nostrand, New Jersey, 1955.

6. O. Ore
   Theory of Graphs, Amer. Math. Soc. Coll. Publ. 38, Amer. Math. Soc., Providense, 1962.